Math 245B Lecture 18 Notes

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1 The Riesz Representation Theorem

1.1 Triangle inequality for complex measures

Lemma 1.1. *Let $(X, \mathcal{M})$ be measurable with complex measures ν_1, ν_2. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.*

Proof. Given ν, find a positive measure $\mu \gg \nu$. Then we get $d\,\nu = f\,d\mu$. Now $d|\nu| = |f|\,d\mu$. Similarly, let $d\nu_i = f_i\,d\mu$ for $\mu = |\nu_1| + |\nu_2|$. Then $d(\nu_1 + \nu_2) = (f_1 + f_2)\,d\mu$, so $d|\nu_1 + \nu_2| = |f_1 + f_2|\,d\mu \leq |f_1|\,d\mu + |f_2|\,d\mu$. $\square$

1.2 Positive linear functionals and the Riesz-Markov theorem

Let (X, ρ) be a compact metric space. The goal is to describe the dual of $C(X, \mathbb{R})$ or $C(X, \mathbb{C}) = C(X)$ with the uniform norm. Recall the Riesz-Markov theorem:

Definition 1.1. A linear functional $\ell : C(X, \mathbb{R}) \to \mathbb{R}$ is **positive** if $\ell(f) \geq 0$ whenever $f \geq 0$.

So if $f \geq g$ then $\ell(f) = \ell(g) + \ell(f - g) \geq \ell(g)$.

Theorem 1.1 (Riesz-Markov)**.** *For any positive linear functional $\ell : C(X, \mathbb{R}) \to \mathbb{R}$, there exists a unique finite positive Borel measure μ on X such that $\ell(f) = \int f\,d\mu$.*

Remark 1.1. If ℓ is a positive linear functional on $C(X, \mathbb{R})$ and $f \in C(X, \mathbb{R})$, then $-\|f\|_u \leq f \leq \|f\|_u$. Then $-\|f\|_u c_x \leq f \leq \|f\|_u c_x$, where c_x is the constant x function. So $-\|f\|_u \ell(c_x) \leq \ell(f) \leq \|f\|_u \ell(c_x)$, which gives $|\ell(f)| \leq \|f\|_u \ell(c_x)$ with equality if $f = c_x$. So $\|\ell\|_{C(X,\mathbb{R})^*} = \ell(c_x) = \mu(X)$.

1.3 The Riesz representation theorem

Let $M(M, K)$ be the space of all finite signed or complex measures on $(X, \mathcal{B}_X)$, where $K = \mathbb{R}$ or $\mathbb{C}$. This is a vector space over K.

Lemma 1.2. *$M(X,K)$ is a normed space over K with norm $\|\mu\| = |\mu|(X)$.*

Proof. If $\lambda \in K$ and $\mu \in M(X,K)$, then $d\mu = f\,d|\mu|$. So $d(\lambda\mu) = (\lambda f)\,d|\mu|$, and we get $d|\lambda\mu| = |\lambda f| d|\mu| = |\lambda|\,d|\mu|$. So $\|\lambda\mu\| = |\lambda|\|\mu\|$.

If $\nu_1, \nu_2 \in M(X,\mathbb{C})$, then by the lemma, we have $\|\nu_1+\nu_2\| = |\nu_1+\nu_2|(X) \leq |\nu_1|(X) + |\nu_2|(X) = \|\nu_1\| + \|\nu_2\|$.

If $\|\nu\| = 0$, then $|\nu|(X) = 0$, so $|\nu| = 0$ by monotonicity. Then $\nu = 0$ because $\nu \ll |\nu|$. $\square$

Theorem 1.2 (Riesz representation)**.** *For $\mu \in M(X,\mathbb{R})$, define $\ell_\mu \in C(X,\mathbb{R})^*$ by $\ell_\mu(f) = \int f\,d\mu$. Then $\mu \mapsto \ell_\mu$ is an isometric isomorphism $M(X,\mathbb{R}) \to C(X,\mathbb{R})^*$. The same holds if we replace $\mathbb{R}$ by $\mathbb{C}$.*

Here is a lemma we will need.

Lemma 1.3. *If $\ell \in C(X,\mathbb{R})^*$, then $\ell = \varphi - \psi$ for some positive linear functionals φ, ψ on $C(X,\mathbb{R})$.*

Proof. For $f \in C(X,\mathbb{R})$ with $f \geq 0$, define $\varphi(f) = \sup\{\ell(g) : 0 \leq g \leq f\}$. For general f define $\varphi(f) := \varphi(f^+) - \varphi(f^-)$. We need to show that φ is a positive linear functional such that $\varphi \geq \ell$. Then we can just define $\psi := \varphi - \ell$. By definition, we have $\varphi(f) \geq \ell(f)$ if $f \geq 0$, which gives us the inequality.

To show that φ is a linear functional, we take a few steps:

1. Suppose $f, h \geq 0$. Then for all $0 \leq g_1 \leq f$ and $0 \leq g_2 \leq h$, we have $0 \leq g_1 + g_2 \leq f + h$. So $\varphi(f+h) \geq \ell(g_1) + \ell(g_2)$ for all such g_1, g_2. Taking the sup over such g_1, g_2, we get $\varphi(f+h) \geq \varphi(f) + \varphi(h)$.

   Conversely, if $0 \leq g \leq f + h$, define $g_1 := \min\{g, f\}$. If $\min = f$ at some x, then $g(x) - g_1(x) = g(x) - f(x) \leq f(x) + h(x) - f(x) \leq h(x)$. So $g_2 := g - g_1 \leq h$. Take the sup over g to get $\varphi(f+h) \leq \varphi(f) + \varphi(h)$. So we get equality.

2. If $f = f^+ - f^- = g - h$ for $g, h \geq 0$, then $f^+ + h = g + f^-$, so step 1 gives $\varphi(f^+) + \varphi(h) = \varphi(g) + \varphi(f^-)$, so $\varphi(f) = \varphi(f^+) - \varphi(f^-) = \varphi(g) - \varphi(h)$.

3. For all f, h, we have $f + h = (f^+ + h^+) - (f^- + h^-)$. So $\varphi(f+h) = \varphi(f^+ + h^+) - \varphi(f^- + h^-) = (\varphi(f^+) - \varphi(f^-)) + (\varphi(h^+) - \varphi(h^-)) = \varphi(f) + \varphi(h)$. So φ is additive.

Similarly, $\varphi(\lambda f) = \lambda\varphi(f)$ for all $\lambda \in \mathbb{R}$. $\square$

Proof. Definitely, $\ell_\mu \in C(X,K)^*$ for all $\mu \in M(X,K)$. Next, suppose $\ell \in C(X,\mathbb{R})^*$. Then $\ell = \varphi - \psi$, where $\varphi, \psi \geq 0$. By Riesz-Markov, we get $\ell = \ell_{\mu_1} - \ell_{\mu_2}$ for some $\mu_1, \mu_2 \geq 0$. So $\ell = \ell_{\mu_1 - \mu_2}$. If $\ell \in C(X,\mathbb{C})$, we can represent this as

$$\ell(f) = \ell_{\mu_1}(\mathrm{Re}(f)) - i\ell_{\mu_2}(i\,\mathrm{Im}(f)) = \ell_{\mu_1 - i\mu_2}(f).$$

It remains to show that $\|\ell_\mu\|$. Let's just prove this for the complex case; the real case is the same argument. We have $\ell_\mu(f) = \int f \, , d\mu$, so we get

$$|\ell_\mu(f)| = |\int f \, d\mu \le \int |f| d|\mu| \le \|f\|_u \int 1 \, d|\mu| = \|f\|_u \cdot \|u\|.$$

Let $d\mu = k \, d|\mu|$, where $k = d\mu/d|\mu|$ is measurable from $X \to S^1$. Now use the fact that for any ε_0, there exists $f \in C(X, \mathbb{C})$ such that $\|f - k\|_{L^1(|\mu|)} < \varepsilon$. We may assume $|f| \le 1$. Now

$$\ell_\mu(\overline{f}) = \int \overline{f} \, d\mu = \int \overline{f} k \, d|\mu| \approx_\varepsilon \int \overline{k} k \, d|\mu| = \int 1 \, d|\mu| = \|\mu\|.$$

So $\|\ell_\mu\| \ge \|\mu\| - \varepsilon$ for all $\varepsilon > 0$. $\square$